Pierre-Simon Laplace (1749-1827)

Laplace Transform

Nikesh Bajaj
nikesh.14730@lpu.co.in
Digital Signal Processing
School of Electronics and Communication
Lovely Professional University

Introduction

- Why
- How

Moving from Fourier to Laplace

- F(jw) = Fourier
- F(s) Laplace
- $F(s) = \mathcal{L}\{f(t)\} \doteq \int_0^\infty e^{-st} f(t)\,dt$ Unilateral Laplace

Example

- x(t) = exp(-at)u(t)
- x(t) = -exp(-at)u(-t)
- x(t) = 3exp(-2t)u(t) -2exp(-at)u(t)
- x(t) =s(t)

Example

$$f(t) = 1,\ 0 \le t < \infty.$$

Pairs

$f(t) = 1,\ t \ge 0$	$F(s) = \frac{1}{s},\ s \ge 0$
$f(t) = t^n,\ t \ge 0$	$F(s) = \frac{n!}{s^{n+1}},\ s \ge 0$
$f(t) = e^{at},\ t \ge 0$	$F(s) = \frac{1}{s-a},\ s > a$
$f(t) = \sin(kt),\ t \ge 0$	$F(s) = \frac{k}{s^2+k^2}$
$f(t) = \cos(kt),\ t \ge 0$	$F(s) = \frac{s}{s^2+k^2}$
$f(t) = \sinh(kt),\ t \ge 0$	$F(s) = \frac{k}{s^2-k^2},\ s > \lvert k \rvert$
$f(t) = \cosh(kt),\ t \ge 0$	$F(s) = \frac{s}{s^2-k^2},\ s > \lvert k \rvert$

Example

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Properties of ROC

PROPERTIES OF THE REGION OF CONVERGENCE

- The ROC contains no poles

$$X(s) = \frac{N(s)}{D(s)}$$

poles of X(s) => D(s) = 0

- The ROC of X(s) consists of a strip parallel to the $j\omega$-axis in the s-plane
- $\mathcal{F}\{x(t)\}$ converges <=> ROC includes the $j\omega$-axis in the s-plane

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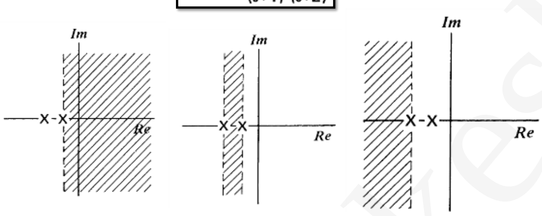
Property 1: ROC does not contain any pole

1. No Poles in ROC

$$X(s) = \frac{1}{(s+1)(s+2)}$$



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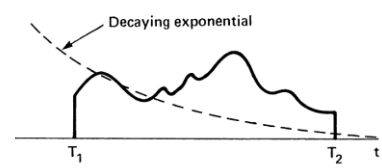
Property 2: for finite duration and absolutely integral signal ROC is entire s-place

2. Finite duration signal

- x(t) finite duration
  => ROC is entire s-plane



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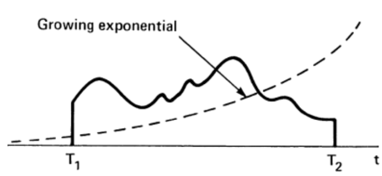
- x(t) finite duration
  => ROC is entire s-plane



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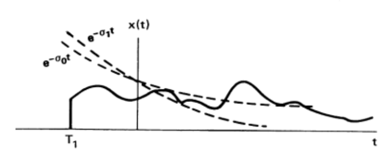
Property 3: for right sided signal if X(s) convergence at p1 and will also converge at p2>p1

3. Right sided signal

x(t) right-sided and $Re\{s\} = \sigma_0$ is in ROC

=> all values for which $Re\{s\} > \sigma_0$ are in ROC



x(t) right-sided and X(s) rational

=> ROC is to the right of the rightmost pole.

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Property 4: for left sided signal if X(s) convergence at p1 and will also converge at p2<p1

4. Left sided signal

- x(t) left-sided and $Re\{s\} = \sigma_o$ is in ROC => all values for which $Re\{s\} < \sigma_o$ are in ROC
- x(t) left-sided and X(s) rational => ROC to the left of the leftmost pole.

Property 5: for two sided signal ROC will be strip

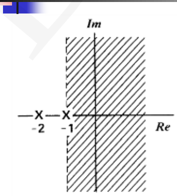
5. Two sided signal

- x(t) two-sided and $Re\{s\} = \sigma_o$ is in ROC => ROC is a strip in the s-plane

Properties

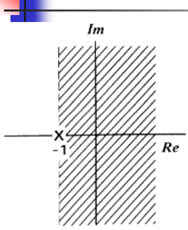
Property 6: ROC is bounded by poles or can extends to infinity,
Property 7: ROC is a strips, parallel to jw axis

Example



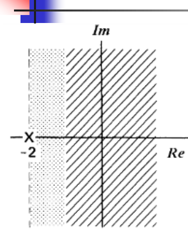
$$X(s) = \frac{1}{(s+1)(s+2)} \qquad Re\{s\} > -1$$

$$= \frac{1}{s+1} - \frac{1}{s+2} \qquad Re\{s\} > -1$$



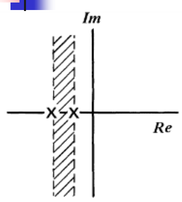
$$X_1(s) = \frac{1}{s+1} \qquad Re\{s\} > -1$$

$$x_1(t) = e^{-t}\,u(t)$$



$$X_2(s) = \frac{-1}{s+2} \qquad Re\{s\} > -1$$

$$x_2(t) = -e^{-2t}\,u(t)$$



$$X(s) = \frac{1}{(s+1)(s+2)} \quad -2 < Re\{s\} < -1$$

$$= \frac{1}{s+1} - \frac{1}{s+2} \quad -2 < Re\{s\} < -1$$

Properties of Laplace

Table 7: **Properties of the Laplace Transform**

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0t}x(t)$	$X(s - s_0)$	Shifted version of R [i.e., s is in the ROC if $(s - s_0)$ is in R]

Properties

Table 7: **Properties of the Laplace Transform**

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Problem

- x(t) =exp(-t)d/dt (exp(-(t+1)u(t+1)))
- x(t) =int[exp(2T)sin(T)u(T) dT]-inf to t

Initial and Final value theorem

Initial- and Final Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Inverse Laplace

- Partial Fraction method